

The Twins Paradox – a Simple Special Relativity Solution

The so-called “Twins paradox” has a very elegant solution via the use of Minkowski diagrams, All measurements in this exercise are from the point of view of the “stay at home” twin. Referring to figure 1, we observe that, the “stay at home” twin measures that the twin in the rocket has flown a time interval:

$$t_{\text{hom}} = \frac{1}{c} \int_0^T ds = \frac{1}{c} \int_0^T \sqrt{(cdt)^2 - dx^2} = \frac{1}{c} \int_0^T \sqrt{(cdt)^2 - 0} = \frac{1}{c} cT = T \quad (1)$$

From figure 1 we notice that the “stay at home” twin measures the same number of clock ticks as the “rocket twin” (the region AB) plus a number of extra clock ticks (the region BC) due to the fact that light signals (represented by the red Minkowski lines inclined at 45°) travel at the finite speed c between the rocket and the Earth. When the rocket turns around at the half way point, the “stay at home” twin will count some time (region BC) that does not exist from the perspective of the “rocket” twin. Therefore the “stay at home” twin would have measured a longer time when he is reunited with his traveling sibling at the end of the trip. The question is: how much extra time? The answer is:

$$\begin{aligned} t_{\text{rocket}} &= \frac{1}{c} \int_0^T ds = \frac{1}{c} \left(\int_{x=vt} ds + \int_{x=-vt} ds \right) = \frac{1}{c} \left(\int_0^{T/2} \sqrt{(cdt)^2 - dx^2} + \int_{T/2}^T \sqrt{(cdt)^2 - dx^2} \right) = \\ &= \frac{1}{c} \left(\int_0^{T/2} \sqrt{(cdt)^2 - (vdt)^2} + \int_{T/2}^T \sqrt{(cdt)^2 - (-vdt)^2} \right) = \frac{\sqrt{c^2 - v^2}}{c} \int_0^T dt = T \sqrt{1 - \frac{v^2}{c^2}} \end{aligned} \quad (2)$$

$$t_{\text{hom}} - t_{\text{rocket}} = T \left(1 - \sqrt{1 - \frac{v^2}{c^2}} \right) \quad (3)$$

For $v \ll c$ we obtain :

$$t_{\text{hom}} - t_{\text{rocket}} = \frac{1}{2} T \frac{v^2}{c^2} \quad (4)$$

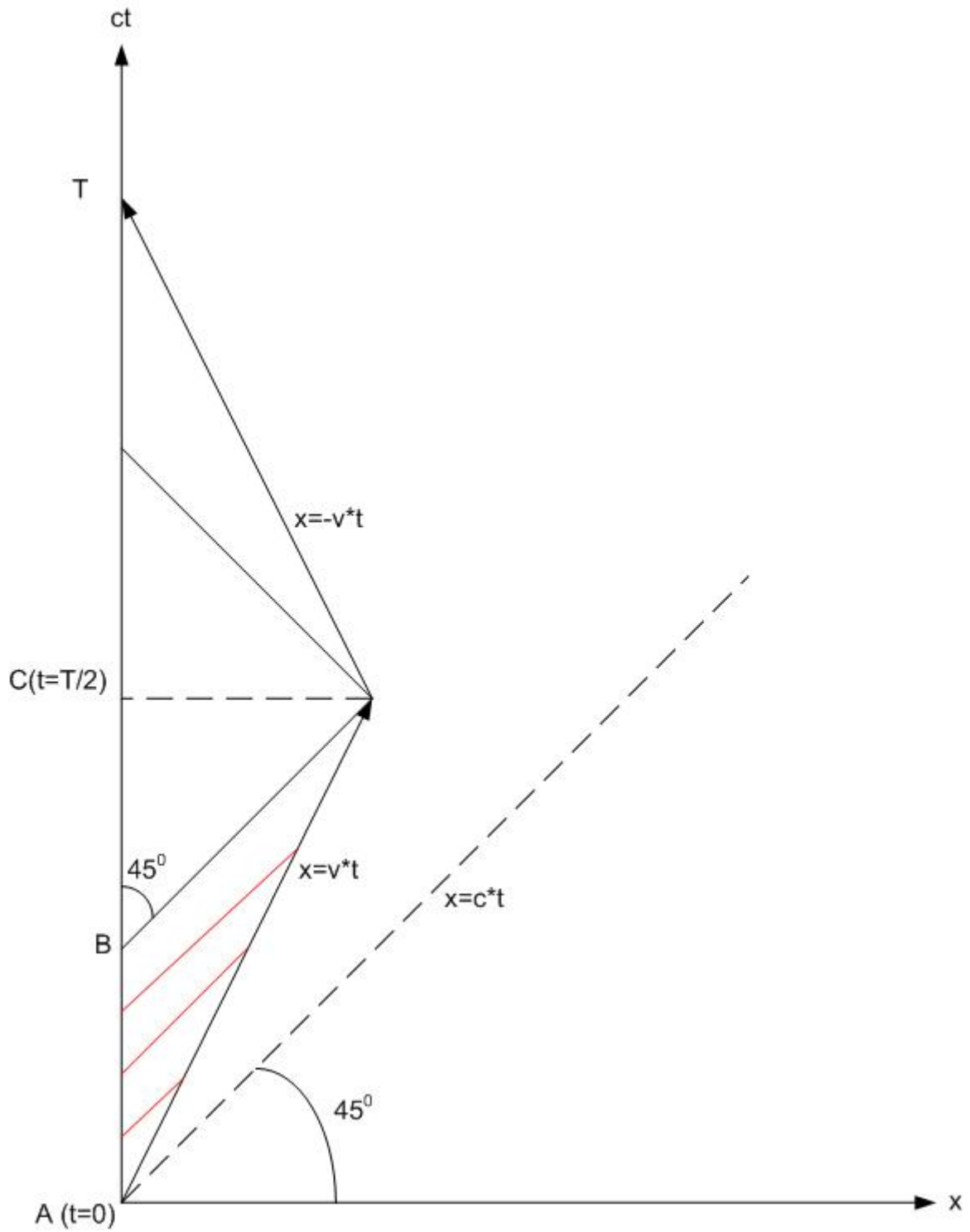


Figure 1 The Minkowski time diagram of the Twin paradox